AN EXPOSITION OF Siddhartha Basak's "Bounds on Factors of Odd Perfect Numbers"

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The paper does not give an example of an Odd Perfect Number, which is not odd because there are articles that argue the non-existence of Odd Perfect Numbers such as Scott Hampton's "Proof of the Non-Existence of Odd Perfect Numbers". However, this paper did not mention any notion of the non-existence of Odd Perfect Numbers but rather provided a lower bound and an upper bound on the factors of an Odd Perfect Number. On this line, it can be said that Siddharta Basak somehow assumes that Odd Perfect Numbers do exist. Another theorist Stephanie Mijat believes that Odd Perfect Numbers may be generated by a different unknown formula. She also revealed that NO ODD PERFECT NUMBERS HAVE YET BEEN DISCOVERED. However, Mijat mentioned that there are characteristics of this non-existent number that have been established. The following is her long list.

- The largest prime factor of an Odd Perfect Number is greater than 10^8, the second largest is 10^4, and the third largest is greater than 100.
- The smallest prime factor of an Odd Perfect Number is less than (2k+8)/3
- An Odd Perfect Number has at least 75 prime factors.
- An Odd Perfect Number is not divisible by 105 and its number of divisors must be even.
- An Odd Perfect Number N is such that N=1(mod12) or N=117(mod468) or N= 81(mod 324)
- Any Odd Perfect Number has at least 7 distinct prime factors.
- If an Odd Perfect Number exists, it must exceed 10^{300} and its largest prime factor must exceed 500,000.
- Any Odd Perfect Number n must be of the form $n = p^{\alpha}m^2$ with p being prime and p$\equiv \alpha \equiv 1(mod4), which\ implies\ that\ n \equiv 1(mod4)$.
- If n is an odd number of the form 6k -1, then n is not perfect.
- Any Odd Perfect Number must have the form 12m+1 or 36m+9.

With this long list of descriptions of an Odd Perfect Number, the possibility of its non-existence is said to be still an open room while number theorists continue the search which is continuing for centuries already. The existence of even perfect numbers is not as mysterious. Euclid gave the first four perfect numbers: 6, 28, 496, and 8128, and they are all even. Perfect numbers are positive integers that is the sum of all its positive divisors, excluding themselves, i.e., 6 = 1 + 2 + 3; 28 = 1+2+4+7+14. Perfect numbers are also defined as positive integers that is half the sum of all its positive divisors including themselves, i.e. $6 = \frac{1+2+3+6}{2}$; $28 = \frac{1+2+4+7+14+28}{2}$. If $\sigma(N) = \sum_{d|n} d$ then a number N is called perfect iff $\sigma(N) =$ 2N, i.e. $\sigma(6) = 1+2+3+6 = 2(6)$; $\sigma(28) = 1+2+4+7+14+28 = 2(28)$. These even perfect numbers can also be generated by Euclid's formula $2^{p-1}(2^p - 1)$ *where p is prime, i.e.* $6 = 2^{2-1}(2^2-1)$; $28 = 2^{3-1}(2^3-1)$. *In February* 2015, *t* he 42nd known perfect number was discovered and all 42 of them are even.

Thus, though the formula to generate an Odd Perfect Number is still unknown, or at least a single one, Siddharta Basak contemplated on the bounds of its supposed factors. Basak utilized the Eulerian form which means that "if a number n is an Odd Perfect Number, then n is of the form $n = p^b q_1^{2a_1} q_2^{2a_2} \dots q_r^{2a_r}$ where p , q_1 , q_1 ... q_r are primes and p $\equiv 1\ (mod 4)$" *or that all but the special prime will have even exponents.*

The two main theorems of his paper are:

Theorem 1: For a positive integer $\alpha \leq h_p$ *and for all p* , $\prod_{p|n} \sum_{i=0}^{\alpha} p^{-i}$ is bounded below by $\frac{2^{\alpha+2}}{\varsigma(\alpha+1)(2^{\alpha+1})}$ where $\varsigma(s) = \prod_{p\ prime} \frac{1}{1-p^{-s}}$ (*the Riemann Zeta Function*) and bounded above by the number 2 .

Equivalently,

$$\frac{2^{\alpha+2}}{\varsigma(\alpha+1)(2^{\alpha+1})} < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} < 2$$

for a positive integer $\alpha \leq h_p$ *for all p where* $\varsigma(s)$ *is the Riemann Zeta Function.*

Theorem 2: If n is an Odd Perfect Number and ω(n) = m is the number of distinct prime factors of n, then there exists primes $p_{I_1},\ p_{I_2}, and\ \ p_{I_3}, such\ that\ the$ first, second, and third prime factors are less than the respective $p_{I_k}s$, where the $p_{I_k}s\ can\ be\ determined$, given m, ($p_r\ is\ the\ r^{th}\ prime$).

Throughout the length of the paper, n is an Odd Perfect Number and p is prime.

PRELIMINARIES

It is known that if n is a perfect number then $\sigma(n),\ thesum\ of\ the\ positive\ divisors\ of\ n\ equals\ 2n$. In this paper, $\sigma(n)\ is\ expressed\ as \prod_{p|n}\sum_{k=0}^{h_p} p^k$= 2n where h_p is the degree of p in the prime factorization of n. This was used to give a proof of a result concerning the sum of the reciprocals of the factors of n, $\sigma_{-1}(n)$:

$$\sigma_{-1}(n) = \prod_{p|n}\sum_{k=0}^{h_p} p^{-k}$$

Proof:

Since n is a perfect number,

$$\sigma(n) = 2n$$

Isolating 2 we have,

$$\frac{\sigma(n)}{n} = 2$$

Substituting the value of $\sigma(n)$,

$$\frac{\prod_{p|n}\sum_{k=0}^{h_p} p^k}{n} = 2$$

Substituting the value of n in the denominator , the left hand side becomes

$$\frac{\prod_{p|n}\sum_{k=0}^{h_p} p^k}{\prod_{p|n} p^{h_p}}$$

which can be simplified to,

$$\prod_{p|n}\frac{\sum_{k=0}^{h_p} p^k}{p^{h_p}}$$

$$= \prod_{p|n}\sum_{k=0}^{h_p}\frac{p^k}{p^{h_p}}$$

$$= \prod_{p|n}\sum_{k=0}^{h_p} p^{k-h_p}$$

$$= \prod_{p|n}\left(p^{0-h_p} + p^{1-h_p} \ldots + p^{h_p-h_p}\right)$$

$$= \prod_{p|n}\left(p^{-h_p+0} + p^{-h_p+1} \ldots + p^{-h_p+h_p}\right)$$

$$=\prod_{p|n}\left(p^{-h_p} + p^{-h_p+1} \ldots + p^0\right)$$

$$=\prod_{p|n}\left(p^0 + \cdots + p^{-h_p+1} + p^{-h_p}\right)$$

$$=\prod_{p|n}\sum_{k=0}^{h_p} p^{-k}$$

Thus it was established that $\prod_{p|n}\sum_{k=0}^{h_p} p^{-k} = 2$ or that the sum of the reciprocals of the factors of n is 2. This well-known result is used as part of a proof for the first theorem.

The main result of the paper is on the bounds for $\prod_{p|n}\sum_{i=0}^{\alpha} p^{-i}$ for a predetermined α. Note The form of this expression is the form of the preliminary result.

Theorem 1:

$$\frac{2^{\alpha+2}}{\varsigma(\alpha+1)(2^{\alpha+1} -)} < \quad \prod_{p|n}\sum_{i=0}^{\alpha} p^{-i} \quad < 2$$

for a positive integer $\alpha \leq h_p$ *for all* p *where* $\varsigma(s)$ *is the Riemann Zeta Function*.

Proof:

From the preliminary result we have,

$$2 = \prod_{p|n} \sum_{k=0}^{h_p} p^{-k}$$

Since $(\alpha+1)\left[\frac{h_p}{\alpha+1}\right]+\alpha \geq h_p$ and substituting h_p with $(\alpha+1)\left[\frac{h_p}{\alpha+1}\right]+\alpha$, we have

$$2 \quad \leq \quad \prod_{p|n} \sum_{k=0}^{(\alpha+1)\left[\frac{h_p}{\alpha+1}\right]+\alpha} p^{-k}$$

The summation of the right hand side will be separated, thus we have

$$\prod_{p|n} \sum_{i=0}^{\alpha} \sum_{j=0}^{\left[\frac{h_p}{\alpha+1}\right]} p^{-(j(\alpha+1)+i)}$$

$$\prod_{p|n} \sum_{i=0}^{\alpha} \sum_{j=0}^{\left[\frac{h_p}{\alpha+1}\right]} p^{-j(\alpha+1)-i}$$

$$\prod_{p|n} \sum_{i=0}^{\alpha} \sum_{j=0}^{\left[\frac{h_p}{\alpha+1}\right]} p^{-j(\alpha+1)}\, p^{-i}$$

$$= \prod_{p|n} \left(\sum_{i=0}^{\alpha} p^{-i} \sum_{j=0}^{\left[\frac{h_p}{\alpha+1}\right]} p^{-j(\alpha+1)} \right)$$

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \prod_{p/n} \sum_{j=0}^{\left[\frac{h_p}{\alpha+1}\right]} p^{-j(\alpha+1)}$$

If j is considered up to infinity, the right hand side will be less than $\prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \prod_{p/n} \sum_{j=0}^{\infty} p^{-j(\alpha+1)}$. Thus we have,

$$2 < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \prod_{p/n} \sum_{j=0}^{\infty} p^{-j(\alpha+1)}$$

Considering only in the summation the addends where p is prime, p not equal to 2, we have

$$2 < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \prod_{p\ prime, p\neq 2} \sum_{j=0}^{\infty} p^{-j(\alpha+1)}$$

Separating p = 2 we have,

$$2 < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\prod_{p\ prime} \sum_{j=0}^{\infty} p^{-j(\alpha+1)}}{\sum_{j=0}^{\infty} 2^{-j(\alpha+1)}}$$

$\sum_{j=0}^{\infty} p^{-j(\alpha+1)}$ is a geometric series that is equivalent to $\frac{1}{1-p^{-(\alpha+1)}}$, similarly $\sum_{j=0}^{\infty} 2^{-j(\alpha+1)} = \frac{1}{1-2^{-(\alpha+1)}}$, thus we have,

$$2 < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\prod_{p\ prime} \frac{1}{1-p^{-(\alpha+1)}}}{\frac{1}{1-2^{-(\alpha+1)}}}$$

Since $\varsigma(s) = \prod_{p\ prime} \frac{1}{1-p^{-s}}$, as defined, the right hand side is

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\varsigma(\alpha+1)}{\frac{1}{1-2^{-(\alpha+1)}}}$$

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\varsigma(\alpha+1)}{\dfrac{1}{1-\frac{1}{2^{\alpha+1}}}}$$

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\varsigma(\alpha+1)}{\dfrac{1}{\frac{2^{\alpha+1}-1}{2^{\alpha+1}}}}$$

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \frac{\varsigma(\alpha+1)}{\frac{2^{\alpha+1}}{2^{\alpha+1}-1}}$$

$$= \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i} \varsigma(\alpha+1) \frac{2^{\alpha+1}-1}{2^{\alpha+1}}$$

Taking $\varsigma(\alpha+1) \frac{2^{\alpha+1}-1}{2^{\alpha+1}}$ out of the operator because they are constants, we have

$$= \varsigma(\alpha+1) \frac{2^{\alpha+1}-1}{2^{\alpha+1}} \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i}$$

This implies that we have

$$2 < \varsigma(\alpha+1) \frac{2^{\alpha+1}-1}{2^{\alpha+1}} \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i}$$

Multiplying both sides of the inequality by $\frac{2^{\alpha+1}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)} > 0$, we have

$$\left(\frac{2^{\alpha+1}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)}\right) 2 < \left(\frac{2^{\alpha+1}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)}\right) \varsigma(\alpha+1) \frac{2^{\alpha+1}-1}{2^{\alpha+1}} \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i}$$

$$\frac{2^{\alpha+1}(2)}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)} < \prod_{p|n} \sum_{i=0}^{\alpha} p^{-i}$$

$$\frac{2^{\alpha+2}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)} < \prod_{p|n}\sum_{i=0}^{\alpha} p^{-i} \quad \text{for some } \alpha \leq \mathrm{h_p}.$$

Thus the lower bound for $\prod_{p|n}\sum_{i=0}^{\alpha} p^{-i}$ is $\frac{2^{\alpha+2}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)}$.

The upper bound is obtained using this argument:

Since in the Eulerian form of an Odd Perfect Number has every exponent (the $\mathrm{h_p}$s) is even other than the first, the expression $\prod_{p|n}\sum_{i=0}^{\alpha} p^{-i}$ *does not have* $\alpha = h_p$ *for all* p. This means $\alpha < \mathrm{h_p}$.

This implies that we can set up the inequality,

$$\frac{2^{\alpha+2}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)} < \prod_{p|n}\sum_{i=0}^{\alpha} p^{-i} < \prod_{p|n}\sum_{i=0}^{h_p} p^{-i}$$

Since it was proven earlier that $\prod_{p|n}\sum_{k=0}^{h_p} p^{-k} = 2$ or equivalently $\prod_{p|n}\sum_{i=0}^{h_p} p^{-i} = 2$

We now finally have

$$\frac{2^{\alpha+2}}{\varsigma(\alpha+1)\,(2^{\alpha+1}-1)} < \prod_{p|n}\sum_{i=0}^{\alpha} p^{-i} < 2.$$

Trying the formula for α = 1 ($h_p \geq 1$), *we have*

$$\frac{2^{1+2}}{\varsigma(1+1)\,(2^{1+1}-1)} = \frac{2^3}{\varsigma(2)(3)} = \frac{8}{3\varsigma(2)} = \frac{16}{\pi^2} \approx 1.621138938 < 2.$$

Note that here, we have verified that α = 1 works. However Basak said that it is not as strong a bound as obtained by placing α = 2 because for α = 2 we have,

$\frac{2^{2+2}}{\varsigma(2+1)\,(2^{2+1}-1)}$ $= \frac{2^4}{\varsigma(3)(7)} = \frac{16}{7\varsigma(2)} \approx$ 1.901502566 *which is closer to* 2. Wolfram Alpha was utilized in the computation.

Since in the Eulerian form of a perfect number, all the other exponents besides the first is even, α = 2 should work; α ≥ 2 should also work. Also since the first factor is p (refer to Eulerian form) is such that p $\equiv$ 1 $mod\ 4$, *two cases have emerged, that is, the case where the exponent b (first exponent) of* p *is* 1, and the other case where b is greater than 2 (i.e. b = 5, 9, …)

Thus for α = 2, we have the lower bound as $\frac{2^{2+2}}{\varsigma(2+1)\,(2^{2+1}-1)}$

For the first case and isolating the first prime, q, we have,

$$\frac{2^{2+2}}{\varsigma(2+1)\,(2^{2+1}-1)} = \frac{2^4}{\varsigma(3)\,(2^3-1)} = \frac{16}{7\varsigma(3)} \approx 1.901502566 < \frac{16q^3}{7\varsigma(3)(q^3-1)} < \left(1+\frac{1}{q}\right)\prod_{p|n,p\neq q}\left(1+\frac{1}{p}+\frac{1}{p^2}\right) < 2$$

Since from the previous argument, α < h_p for all p.

The second case is simply α= 2 without isolating the first prime q, thus we have

$$\frac{2^{2+2}}{\varsigma(2+1)\,(2^{2+1}-1)} = \frac{2^4}{\varsigma(3)\,(2^3-1)} = \frac{16}{7\varsigma(3)} \approx 1.901502566 < \prod_{p|n,}\left(1+\frac{1}{p}+\frac{1}{p^2}\right) < 2$$

Now, though, nobody in the world has not found an Odd Perfect Number yet, Basak argued that the above provides an efficient, though not optimal method for proving a number is not an Odd Perfect Number. This is because one does not need to know the exponents of the primes or one does not need to obtain a complete factorization of the number. These two cases also supports why all three of 3, 5, and 7 cannot divide an Odd Perfect Number, to show we have,

$$\left(1+\frac{1}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{5}+\frac{1}{5^2}\right)\left(1+\frac{1}{7}+\frac{1}{7^2}\right) \approx 2.083537 > 2.$$

However, it can also be shown that 3 and 5 can divide an Odd Perfect Number with the above case since

$\left(1+\frac{1}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{5}+\frac{1}{5^2}\right) \approx 1.791111 < 2.$

The second main result of the paper is stated as follows:

Theorem 2: If n is an Odd Perfect Number and ω(n) = m is the number of distinct prime factors of n, then there exists primes $p_{I_1},\ p_{I_2}, and\ \ p_{I_3}, such\ that\ the$ first, second, and third prime factors are less than the respective $p_{I_k}s$, where the $p_{I_k}s\ can\ be\ determined$, given m, ($p_r\ is\ the\ r^{th}\ prime$).

The proof espoused is as follows:

Let n be an arbitrary odd perfect number.

Since n is a number, it is the product of unique primes raised to some positive exponents,

$$n = p_{i_1}^{a_1} p_{i_2}^{a_2} \dots p_{i_m}^{a_m}$$

Rewriting the above expression using the results we have in the first theorem, we have

$$n = \prod_{p_{ij}\ |\ n, j=0}^{m} p_{ij}^{-j}$$

$$n = \prod_{p_{ij}\ |\ n, j=1}^{m} \left(1+\frac{1}{p_{ij}}\right)$$

Now , if $p_r^{(1)} is\ defined\ as\ \prod_{j=r}^{r+m-1}\left(1+\frac{1}{p_j}\right)$

Then $p_{i_1}^{(1)} \geq \prod_{p_{ij} \mid n, j=1}^{m} \left(1+\frac{1}{p_{ij}}\right)$. But note that $\lim_{r\to\infty} p_r^{(1)} = \lim_{r\to\infty} \prod_{j=r}^{r+m-1} \left(1+\frac{1}{p_j}\right) = 1 < \frac{16}{\pi^2}$

Therefore there exists I_1 $such\ that\ p_r^{(1)} < \frac{16}{\pi^2} \quad for\ all\ r > I_1$.

Thus if n is an Odd Perfect Number we have $i_1 \leq I_1\ and\ p_{i_1} \leq p_{I_1}$. This proves the existence of p_{I_1}.

Similarly, if $p_r^{(2)} is\ defined\ as\ \left(1+\frac{1}{3}\right) \prod_{j=r}^{r+m-2} \left(1+\frac{1}{p_j}\right)$

Then $p_{i_2}^{(2)} \geq \left(1+\frac{1}{p_{i_1}}\right) \prod_{p_{ij} \mid n, j=2}^{m} \left(1+\frac{1}{p_{ij}}\right)$. Note also that $\lim_{r\to\infty} p_r^{(2)} = \left(1+\frac{1}{3}\right) = \frac{4}{3} < \frac{16}{\pi^2}$.

Therefore there exists I_2 $such\ that\ p_r^{(2)} < \frac{16}{\pi^2} \quad for\ all\ r > I_2$.

Thus if n is an Odd Perfect Number, we have $i_2 \leq I_2\ and\ p_{i_2} \leq p_{I_{2.}}$. This proves the existence of p_{I_2}.

And lastly, $p_r^{(3)} is\ defined\ as\ \left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right) \prod_{j=r}^{r+m-3} \left(1+\frac{1}{p_j}\right)$.

Then $p_{i_3}^{(3)} \geq \left(1+\frac{1}{p_{i_1}}\right)\left(1+\frac{1}{p_{i_2}}\right) \prod_{p_{ij} \mid n, j=3}^{m} \left(1+\frac{1}{p_{ij}}\right)$.

But note that $\lim_{r\to\infty} p_r^{(3)} = \lim_{r\to\infty} \prod_{j=r}^{r+m-3} \left(1+\frac{1}{p_j}\right) = \left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right) = \left(\frac{4}{3}\right)\left(\frac{6}{5}\right) = \frac{8}{5} < \frac{16}{\pi^2}$

Therefore there exists I_3 $such\ that\ p_r^{(3)} < \frac{16}{\pi^2} \quad for\ all\ r > I_3$.

Thus if n is an Odd Perfect Number we have $i_3 \leq I_3\ and\ p_{i_3} \leq p_{I_3}$. This proves the existence of p_{I_3}.

Thus there exists primes $p_{I_1},\ p_{I_2}, and\ p_{I_3}, such\ that\ the$ first, second, and third prime factors are less than the respective $p_{I_k}s$, where the $p_{I_k}s\ can\ be\ determined$, given m, ($p_r\ is\ the\ r^{th}\ prime$).

Note that the above method is very restricted and do not apply for other prime factors since

$\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right) > \frac{16}{\pi^2}$.

Note that even if n is not determined, m , the number of distinct prime factors of n can be specified. Basak gave a table of the values of p_{I_k} for each m=9 up to m=20. The table begins with m=9 since if n is an Odd Perfect Number, it was established in other studies that m= $\omega(n) \geq 9$.

	p_{I_k} (for α = 1)		
M	k=1	k=2	k=3
9	11	31	509
10	11	31	593
11	11	37	659
12	13	41	739
13	13	43	811
14	13	43	881
15	13	47	947
16	13	53	1031
17	17	53	1093
18	17	59	1171
19	17	61	1237
20	17	61	1301

The table above can be read this way: If n is an Odd Perfect Number and if there are m=9 distinct prime factors of n, then there exists primes $p_{I_1},\ p_{I_2}, and\ p_{I_3}, such\ that\ the$ first, second, and third prime factors are less than the respective $p_{I_1} = 11,\ p_{I_2} = 31,\ p_{I_3} = 509$, respectively, and so on.

References:

Charles Greathouse, "Bounding the Factors of Odd Perfect Numbers"

Stephanie Mijat, "Odd Perfect Numbers"

Scott Hampton, " Proof of the Non-Existence of Odd Perfect Numbers "

Siddhartha Basak, "Bounds on Factors of Odd Perfect Numbers"